Example. Solve by conversion to Polar Coordinates

Locate the minimum of the function $f(x, y) = x^2 + y^2$ subject to $3x + 4y = 10$

The original problem is given in Cartesian/rectangular coordinates and can be solved without any conversions. However, the point here is to show how polar coordinates work and how they might simplify the problem somewhat. Let us start by rewriting the objective function by setting the following transformations

$$r \cdot \cos \theta = x$$
$$r \cdot \sin \theta = y,$$

where $r$ stands for the radius, and $\theta$ stands for the angle as shown on the graph below.

Fig. 1 Polar and Cartesian Coordinates

Squaring both sides of the 2 equations, we get

$$r^2 \cos^2 \theta = x^2$$
$$r^2 \sin^2 \theta = y^2$$

Just a quick note on the notation to avoid any ambiguities

$$\sin^2 \theta = (\sin \theta)^2 \text{ \ but not to \ } \sin(\theta^2)$$

Going back to the set of 2 equations and combining them

$$r^2 \cos^2 \theta + r^2 \sin^2 \theta = x^2 + y^2$$

Factoring $r^2$ and using the trigonometric identity $\cos^2 \theta + \sin^2 \theta = 1$:

$$r^2(\cos^2 \theta + \sin^2 \theta) = r^2$$
To sum up, we get:

\[ r^2 = x^2 + y^2 \]

Substituting back in the objective function:

\[ f(r, \theta) = r^2 \]

Now we have to apply the transformation for the constraint as well:

\[ 3x + 4y = 10 \quad \text{Constraint in Rectangular Coordinates} \]

\[ 3r \cdot \cos(\theta) + 4r \cdot \sin(\theta) = 10 \quad \text{Constraint in Polar Coordinates} \]

The assumption that I will make at this point is that the optimum lies at the boundary and hence the constraint is binding. Let us use the new constraint and rewrite it slightly by factoring and solving for the radius \( r \)

\[ r = \frac{10}{3\cos(\theta) + 4\sin(\theta)} \]

I want to find the critical point of the denominator and will do so by setting it equal to some variable \( R \) and differentiating it with respect to the angle \( \theta \).

\[ R = 3\cos(\theta) + 4\sin(\theta) \]

\[ \frac{dR}{d\theta} = -3\sin(\theta) + 4\cos(\theta) = 0 \]

Solving for the angle \( \theta \)

\[ 3\sin(\theta) = 4\cos(\theta) \]

\[ \frac{3\sin(\theta)}{4\cos(\theta)} = 1 \]

Simplifying by using the definition of tangent

\[ \tan(\theta) = \frac{4}{3} \]

Using the inverse tangent to solve for the angle

\[ \tan^{-1}\left(\frac{4}{3}\right) = \theta \]

Plugging this into a calculator and rounding to 2 places, we get
\[ \theta = 53.13 \text{ Degrees} \]

Going back to the constraint and plugging the value of the angle

\[ r = \frac{10}{3\cos(53.13) + 4\sin(53.13)} = \frac{10}{1.8 + 3.2} = 2 \]

The value of the radius at the boundary is 2. Plugging it into the objective function to get the minimum

\[ f_{\text{min}}(2, 53.13^\circ) = 4 \]

Here the answer is defined by a radius and an angle which cannot be interpreted directly in economic terms (only the value of the function can). Now we have to use the reversing transformations (from polar to rectangular) to get the answer in rectangular coordinates where \( x \) and \( y \) might represent some goods, or the capital and labor inputs.

\[ r = \sqrt{x^2 + y^2} \]

\[ \tan(\theta) = \frac{y}{x} \]

Let us start with the second equation since we already now the LHS. Recall

\[ \tan(\theta) = \frac{4}{3} \]

Hence, we know the ratio between the 2 coordinates or goods if you will.

\[ x = \frac{3}{4} y \]

Now using this equality and into the radius equation and substituting for \( x \)

\[ r = 2 = \sqrt{\left(\frac{3}{4}y\right)^2 + y^2} \]

Solving for the \( y \)

\[ 4 = \frac{9}{16} y^2 + y^2 \]

\[ y^2 \left(\frac{9}{16} + 1\right) = 4 \]

\[ y^2 = 2.56 \]
Taking square roots on both sides

\[ y = \pm 1.6 \]

Now using this equation, we find the values of \( x \) at the minimum

\[ x = \frac{3}{4}(\pm 1.6) \]

\[ x = \pm 1.2 \]

Although we get 2 values for \( x \) and \( y \), the minimum occurs only at the positive root. The negative roots will not satisfy the constraint

\[ 3x + 4y = 10 \]

hence we can ignore them in this case.

To summarize the optimal solution of this constrained optimization problem in both Rectangular and Polar coordinates

\[ f_{\text{min}}(2, 53.13°) = 4 \quad \text{Polar Coordinates} \]

\[ f_{\text{min}}(1.2, 1.6) = 4 \quad \text{Rectangular Coordinates} \]

Below is a graphical representation of the problem in Rectangular coordinates.

![Fig. 2 Contour Map in R.C.](image-url)
The constraint is represented by the dotted line and the value of the function itself by the color feature. As we increase the number of contour curves, it will become tangent to a contour curve with value of 4. Contour maps are a great way to visualize 3 dimensional output and can be quite handy when analyzing the behaviour of some function.

A Geometric Shortcut
Actually we can localize the minimum of this function and its constraint and solve the problem without even taking one derivative. The only tools we need are some basic facts about parallel and perpendicular lines. Let us recall the objective function and the constraint.

\[
\begin{align*}
\min f(x, y) &= x^2 + y^2 \quad \text{subject to} \quad 3x + 4y = 10 \implies y = \frac{10}{4} - \frac{3}{4}x
\end{align*}
\]

The objective function is centered at the origin since for \(x, y = 0 \implies f = 0\) and is symmetric. Since this is a minimization problem, in this case we want to find the shortest line connecting the origin and the constraint. Now we need to use the idea of parallel and perpendicular lines. All the information of the constraint is given in the problem. We need to find the equation of the perpendicular red line and find where it intersects the constraint. This is actually quite easy since we already know where it intersects the y-axis. Hence the only missing element is the slope.

![Fig. 3 Parallel and Perpendicular Lines](image)

The lines form a 90 degree angle if their slopes are negative inverses of one another.
We already know the slope of the constraint

\[ \left(-\frac{3}{4}\right) \text{ constraint slope} \]

Its negative inverse will be

\[ -\left(-\frac{3}{4}\right)^{-1} = \frac{4}{3} \]

Now we have all the information needed to construct the equation of the perpendicular red line. Its y-intercept is 0 and we found the slope to be \( \frac{4}{3} \). Hence, the equation is

\[ y_{\text{Red}} = \frac{4}{3}x + 0 = \frac{4}{3}x \]

The last step is to set equal the constraint and the perpendicular line equations in order to find their intercept which will give us the coordinates where the minimum value of the function occurs. Recalling the equation of the constraint from the beginning

\[ y = \frac{10}{4} - \frac{3}{4}x \]

Setting them equal and solving for \( x \)

\[ \frac{4}{3}x = \frac{10}{4} - \frac{3}{4}x \]

After simplifying, we get

\[ x = 1.2 \]

Plugging this result into one of 2 equations and solving for \( y \)

\[ y = \frac{4}{3} \times (1.2) = 1.6 \]

To sum up the results:

\[ \begin{cases} x = 1.2 \\ y = 1.6 \\ f_{\text{min}} = 4 \end{cases} \]

We get the exact same results but in a much faster and simpler way without even working with the objective function. However, it has to be admitted that if we had an asymmetrical function, the problem will become slightly harder.